

## On the extra mode and inconsistency of Hořava gravity

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## On the extra mode and inconsistency of Hořava gravity

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**ABSTRACT:** We address the consistency of Hořava’s proposal for a theory of quantum gravity from the low-energy perspective. We uncover the additional scalar degree of freedom arising from the explicit breaking of the general covariance and study its properties. The analysis is performed both in the original formulation of the theory and in the Stückelberg picture. A peculiarity of the new mode is that it satisfies an equation of motion that is of first order in time derivatives. At linear level the mode is manifest only around spatially inhomogeneous and time-dependent backgrounds. We find two serious problems associated with this mode. First, the mode develops very fast exponential instabilities at short distances. Second, it becomes strongly coupled at an extremely low cutoff scale. We also discuss the “projectable” version of Hořava’s proposal and argue that this version can be understood as a certain limit of the ghost condensate model. The theory is still problematic since the additional field generically forms caustics and, again, has a very low strong coupling scale. We clarify some subtleties that arise in the application of the Stückelberg formalism to Hořava’s model due to its non-relativistic nature.

**KEYWORDS:** Models of Quantum Gravity, Space-Time Symmetries, Renormalization Group

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## 1 Introduction and summary

Recently, Hořava has proposed a new approach to the theory of quantum gravity [1]. The key idea of the proposal is to equip space-time with a new structure: a foliation by space-like surfaces. This foliation defines the splitting of the coordinates into “space” and “time” and breaks the general covariance of general relativity (GR). Then one can improve the UV behavior of the graviton propagator and ultimately make the theory power-counting renormalizable by adding to the GR action terms with higher *spatial* derivatives. At the same time the action in the ADM formalism contains only first order time derivatives, which allows to circumvent the problems with the ghosts appearing in covariant higher order gravity theories [2]. The higher derivative terms naively become irrelevant in the infrared and it was argued in [1] that the theory reduces to GR at large distances.

However, the consistency of the above proposal is far from being clear. The main concern comes from the fact that the introduction of a preferred foliation explicitly breaks the gauge group of GR down to the group of space-time diffeomorphisms preserving this foliation. As already pointed out in [1] this breaking is expected to introduce extra degrees of freedom compared to GR. The new degrees of freedom can persist down to the infrared and lead to various pathologies (instabilities, strong coupling) that may invalidate the theory. An illustration of this phenomenon is provided by theories of massive gravity where special care is needed to make the additional degrees of freedom well-behaved [3–5].

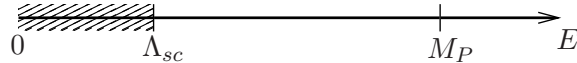
In the recent works in the topic there have been several controversial claims about the properties of the extra degrees of freedom. In [6] the new mode was identified among

the perturbations around a static spatially homogeneous background in the presence of matter. The mode was argued to be strongly coupled to matter in the limit when the theory is expected to approach GR, making it hard to believe that a GR limit exists. It is worth noting that the mode found in [6] is not propagating: its equation of motion does not contain time derivatives [7]. Thus it remains unclear from this analysis whether this mode corresponds to a real degree of freedom or can be integrated out as unphysical. The observation that the extra mode is non-propagating was generalized in [8] to the case of cosmological backgrounds. The interpretation of this result given in [8] is that actually the Hořava gravity is free from additional degrees of freedom. It was also claimed that the strong coupling is alleviated by the expansion of the Universe. Finally, the *non-linear* Hamiltonian analysis performed in [9] shows that the phase space of Hořava gravity is 5-dimensional. This result is puzzling: a normal degree of freedom corresponds to a 2-dimensional phase space; so the result of [9] suggests that the number of degrees of freedom in Hořava gravity is two and a half. Two of these degrees of freedom are naturally identified with the two helicities of graviton. But the physical meaning of the extra “half-mode” is obscure.

The aim of the present paper is to clarify this issue. We show that Hořava gravity does possess an additional light scalar mode. For a general background the equation of motion of this mode contains time derivatives implying that the mode is propagating. The peculiarity of Hořava gravity is that the equation for the extra mode is *first order* in time derivatives. Still, the solution corresponds to waves with a background dependent dispersion relation and is fixed once a single function of spatial coordinates is determined as the initial condition in the Cauchy problem. This explains why this mode corresponds to a single direction in the phase space.

Next we address the consistency of the Hořava proposal by studying the infrared properties of the extra mode. We find that its dynamics exhibits a number of bad features. First, the mode becomes singular for static or spatially homogeneous backgrounds. Namely, the mode frequency diverges in that limit. This explains why this mode has been overlooked in the previous analyses of perturbations in Hořava gravity [6–8]. Second, for certain (background-dependent) values of spatial momentum the mode becomes unstable. Again, the rate of the instability diverges if one takes the static / spatially homogeneous limit for the background metric. Third, we show that at energies above a certain scale the extra mode is strongly coupled to itself, and not only to matter. We find that the strong coupling scale is background dependent and goes to zero for flat / cosmological backgrounds. Hence, the model suffers from a much more severe strong coupling problem than pointed out in [6], where the dependence of the strong coupling scale on the background curvature was ignored. Because of the strong coupling the Hořava model can be trusted only in a narrow window of very small energies, way below the Planck scale; this point is illustrated schematically in figure 1. This implies that the Hořava model cannot be considered as consistent theory of quantum gravity.

To unveil the properties of the extra mode we make use of the Stückelberg formalism. For the case at hand the Stückelberg trick is synonymous to the covariantization of the model. As a result we obtain a scalar-tensor theory with the time derivative of the



**Figure 1.** The energy scales appearing in the Hořava model: the Planck mass  $M_P$  and the strong coupling scale  $\Lambda_{sc}$ . The theory can be trusted only in the narrow window of energies  $0 < E < \Lambda_{sc}$  (dashed region). The scale  $\Lambda_{sc}$  is much smaller than  $M_P$ , it depends on the background curvature and goes to zero for flat / cosmological backgrounds, see eq. (5.8) below.

scalar field developing non-zero vacuum expectation value. The invariance under foliation preserving diffeomorphisms implies that the theory has an internal symmetry consisting in reparameterizations of the scalar. We clarify the subtleties that arise in the application of the Stückelberg procedure to Hořava gravity due to the intrinsically non-relativistic nature of the proposal. The covariantization of the higher space derivatives of the model leads to higher covariant derivative operators in the equations of motion. Naively, this would imply the appearance of too many degrees of freedom. However, in the theory at hand the higher derivative operators are of a special type that allows for a well-posed Cauchy problem with reduced number of initial data in the preferred foliation. In this way the number of degrees of freedom is decreased and matches with the number of modes in the non-covariant formulation. As a byproduct, we point out a large class of covariant higher derivative operators that allow for the reduction of the number of degrees of freedom in a preferred Cauchy slicing.

There exist two versions of the original Hořava proposal [1]. The difference between them lies in an additional restriction which can be imposed on the lapse function. Namely, one can require the lapse to be “projectable”, i.e. be constant along the foliation surfaces. In the present paper we are mainly interested in the non-projectable case; all the previous discussion refers to this case. The projectable version of the theory is briefly discussed at the end of the paper. We argue that in this case the Hořava gravity is equivalent to a specific limit of the ghost condensate model [10]. This implies that now the theory possesses a full-fledged extra scalar degree of freedom with second order equation of motion. The classical dynamics of linear perturbations of the scalar is regular for all backgrounds. In this sense the projectable version of the theory is better behaved than the non-projectable one. However, beyond the linear or classical level the additional scalar still exhibits pathologies. As shown in [11], at the classical level the dynamics of the projectable version of Hořava gravity is equivalent to GR supplemented by a pressureless fluid. As we discuss, the fluid component is precisely described by the extra scalar, the fluid velocity being proportional to the scalar gradient. A well-known property of pressureless fluid is to develop caustics where the fluid velocity becomes ill-defined. For the scalar at hand it means that the theory inevitably breaks down after finite amount of time evolution. At the quantum level, the extra mode exhibits unacceptably low scale of strong coupling. We comment on the possible ways to address these problems.

The paper is organized as follows. In section 2 we describe the model and formulate the Cauchy problem for it. In section 3 we derive the linearized equations for perturbations about an arbitrary background. We find an explicit expression for the extra mode and show

that it obeys first order equation in time. In section 4 we turn to the Stückelberg analysis of the model which allows us to study the properties of the extra mode in a transparent way. We discuss the subtleties in the application of the Stückelberg formalism to the case at hand. In section 5 we take the limit when the GR part decouples from the Stückelberg sector and concentrate on the latter. We find that the linear perturbations of the field exhibit fast exponential instability and unacceptably low scale of strong coupling. Finally, in section 6 we consider the projectable version of Hořava gravity and argue that it is equivalent to a specific limit of ghost condensate model. Section 7 contains concluding remarks and discussion of future directions. Technical details are deferred to appendix A.

## 2 Cauchy problem for Hořava gravity

We consider the class of non-relativistic generalizations of GR proposed in [1]. One starts with the ADM decomposition of the space-time metric,

$$ds^2 = (N^2 - N_i N^i) dt^2 - 2N_i dx^i dt - \gamma_{ij} dx^i dx^j .$$

Then the action for a theory of this class can be written in the generic form

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{\gamma} N (K_{ij} K^{ij} - \lambda K^2 + \xi R + \zeta R_{ij} R^{ij} + \dots) , \quad (2.1)$$

where  $M_P$  is the Planck mass;  $K_{ij}$  is the extrinsic curvature tensor for the surfaces of constant time,  $K$  is its trace;  $R_{ij}$ ,  $R$ ,  $\gamma$  are the Ricci tensor, Ricci scalar and the determinant of the spatial metric  $\gamma_{ij}$ , and  $N$  is the lapse function. The extrinsic curvature is related to the time derivative of the metric in the usual way,

$$K_{ij} = \frac{1}{2N} (\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i) . \quad (2.2)$$

Throughout the paper, if not stated otherwise, 3-dimensional indices  $i, j, \dots$  are raised and lowered using  $\gamma_{ij}$ , and the covariant derivatives carrying these indices are understood in the 3-dimensional sense. The ellipsis in (2.1) represents higher order terms constructed out of the metric  $\gamma_{ij}$  using only spatial derivatives and invariant under 3-dimensional diffeomorphisms. As discussed in [1], the introduction of terms of sufficiently high order (with six spatial derivatives) yields a theory which is naïvely power-counting renormalizable.

The purpose of the present paper is to study the properties of the theory (2.1) in the infrared. The precise structure of the higher order terms is not essential for this analysis. To illustrate the general effect of these terms it suffices to consider explicitly one of them which we choose to be the square of the Ricci tensor.

The action (2.1) reduces to GR for the values of the parameters<sup>1</sup>  $\lambda = 1$ ,  $\zeta = 0$ . Away from these values (2.1) explicitly breaks general covariance down to the subgroup consisting of spatial diffeomorphisms and reparameterizations of time,

$$\mathbf{x} \mapsto \tilde{\mathbf{x}}(t, \mathbf{x}), \quad t \mapsto \tilde{t}(t) . \quad (2.3)$$

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<sup>1</sup>The value of the parameter  $\xi$  is not important: it can always be absorbed into the rescaling of the time coordinate and the shift  $N_i$ .

These transformations preserve the foliation of the space-time by the surfaces  $t = \text{const.}$  Note that if one assumes that the action (2.1) defines a consistent quantum theory, one expects all the parameters in this action to acquire radiative corrections and flow by the renormalization group. In particular, the parameter  $\lambda$  is expected to be generally different from 1. It was argued [1] that in the infrared limit the higher order terms become negligible and so one may expect to recover GR if the parameter  $\lambda$  flows to 1 in that limit. We will see below that this expectation is incorrect: the explicit breaking of general covariance leads to the appearance of an extra degree of freedom in the infrared which becomes strongly coupled<sup>2</sup> when  $\lambda$  approaches 1.

Varying the action with respect to  $N$ ,  $N_i$ ,  $\gamma_{ij}$  yields the following equations,

$$-K_{ij}K^{ij} + \lambda K^2 + \xi R + \zeta R_{ij}R^{ij} = 0, \quad (2.4)$$

$$\nabla_i K^{ij} - \lambda \nabla^j K = 0, \quad (2.5)$$

$$\begin{aligned} -\frac{\partial}{\partial t}(K^{ij} - \lambda \gamma^{ij} K) - (1 - 2\lambda)NKK^{ij} - 2NK^{ik}K_k^j + NK_{kl}K^{kl}\gamma^{ij} - \xi NR^{ij} \\ - \xi \gamma^{ij} \Delta N + \xi \nabla^i \nabla^j N - 2\zeta NR^{ik}R_k^j - \zeta \nabla_k \nabla_l (NR^{kl})\gamma^{ij} - \zeta \Delta (NR^{ij}) \\ + \zeta [\nabla_k \nabla^i (NR^{kj}) + \nabla_k \nabla^j (NR^{ki})] = 0. \end{aligned} \quad (2.6)$$

Here  $\Delta \equiv \gamma^{ij} \nabla_i \nabla_j$  and we have fixed the gauge  $N_i = 0$ . This system has to be supplemented by the evolution equation (2.2) for the metric which in the chosen gauge takes the form

$$\dot{\gamma}_{ij} = 2NK_{ij}. \quad (2.7)$$

Let us analyze the Cauchy problem for the system (2.4)–(2.7). The set of initial data at  $t = 0$  consists of the values for  $\gamma_{ij}$ ,  $K^{ij}$  and  $N$  at this time. The initial data have to satisfy the constraints (2.4) and (2.5). Then eqs. (2.6), (2.7) describe the time evolution of the extrinsic curvature and the metric. However, the system (2.4)–(2.7) is incomplete: it does not allow to determine the time evolution of the lapse  $N$ . In GR this ambiguity is a gauge artifact removed by appropriate gauge fixing, e.g.  $N = 1$ . In our case the gauge freedom is absent, and  $N$  is a genuine dynamical field. To obtain the missing equation one notices that, due to the lack of gauge invariance, the Hamiltonian constraint (2.4) is not automatically preserved by the time evolution. Imposing that the constraint holds at any time one gets a secondary constraint which produces an equation for  $N$ . Taking the time derivative of (2.4) and simplifying the result with the use of the rest of the equations we obtain

$$\begin{aligned} \nabla_i \left\{ N^2 \left[ \xi (\lambda - 1) \nabla^i K + \zeta \left( K_{kj} \nabla^i R^{kj} - R^{kj} \nabla^i K_{kj} + K \nabla_j R^{ij} - R^{ij} \nabla_j K \right. \right. \right. \\ \left. \left. \left. - K_{kj} \nabla^k R^{ij} + R^{kj} \nabla_k K_j^i - K_j^i \nabla_k R^{kj} + R^{ij} \nabla^k K_{kj} \right) \right] \right\} = 0. \end{aligned} \quad (2.8)$$

As expected, the l.h.s. of this equation vanishes identically in the case of GR as a consequence of gauge invariance. For  $\lambda$  and  $\zeta$  away from their GR values this equation allows

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<sup>2</sup>More precisely, the additional mode is weakly coupled only in a narrow window at low energies. This window depends both on  $\lambda$  and the parameters of the background geometry; it shrinks to zero both when  $\lambda \rightarrow 1$  or when the background curvature vanishes, see section 5.

to determine the lapse at any moment of time provided the configuration of  $\gamma_{ij}$ ,  $K^{ij}$  is given; in this way it imposes additional constraint on the initial data. It is important to notice that the constraint vanishes whenever the extrinsic curvature or gradients of the curvature tensors are zero. In particular, this happens for spatially homogeneous or static configurations. Note also that eq. (2.8) has the form of the conservation of a (space-like) current. This fact acquires a natural interpretation in the covariant picture where (2.8) becomes the equation of motion of the Stückelberg field, and the current corresponds to the shift symmetry of this field, cf. eq. (4.16) below.

The system (2.4)–(2.8) constitutes the complete set of equations of motion for Hořava gravity. Let us count the number of independent Cauchy data for this system. Originally the set of initial data for  $\gamma_{ij}$ ,  $K^{ij}$ ,  $N$  contains  $6 + 6 + 1 = 13$  functions of spatial variables  $\mathbf{x}$ . The constraints (2.4), (2.5), (2.8) eliminate  $1 + 3 + 1 = 5$  of them. Additionally, 3 functions are removed by the residual (time-independent) gauge transformations of spatial coordinates. Thus we are left with  $13 - 5 - 3 = 5$  arbitrary functions as initial data. 4 of these functions are identified as initial data for the two helicities of the graviton. The remaining freedom in the choice of one more function implies the presence of an extra mode which is absent in GR. Note that because the Cauchy data for this mode are limited to a single function, the corresponding evolution equation must be first order in time. This agrees with the observation made in [9] that the phase space of the Hořava gravity has odd dimensionality. Our task below is to investigate the properties of the extra mode.

### 3 The elusive mode

In this section we reveal the extra mode explicitly. For the sake of the argument we restrict to the case when the higher order terms in the action are absent,  $\zeta = 0$ , and breaking of general covariance arises only from  $\lambda \neq 1$ . This restriction allows to capture the essential physics of the extra mode, without overloading the paper with lengthy formulae. As it will become transparent below, the main conclusions are unaffected by the values of  $\zeta$  and  $\lambda$ , so long as the general covariance is broken by some of these terms. For  $\zeta = 0$  the secondary constraint (2.8) reduces to

$$\nabla_i (N^2 \nabla^i K) = 0. \tag{3.1}$$

Consider small perturbations of the fields about a background  $\bar{\gamma}_{ij}$ ,  $\bar{K}^{ij}$ ,  $\bar{N}$ . We assume that the background satisfies the equations of motion of the Hořava gravity, but is arbitrary otherwise. Thus we write

$$\begin{aligned} \gamma_{ij} &= \bar{\gamma}_{ij} + h_{ij} , \\ K^{ij} &= \bar{K}^{ij} + \kappa^{ij} , \\ N &= \bar{N} + n . \end{aligned}$$



The next step is to plug these expressions into eqs. (2.4) – (2.8) and expand them to linear order in the perturbations  $h_{ij}$ ,  $\kappa^{ij}$ ,  $n$ . We obtain

$$-2\bar{K}_{ij}\kappa^{ij} - 2\bar{K}_k^i\bar{K}^{jk}h_{ij} + 2\lambda\bar{K}\kappa + 2\lambda\bar{K}\bar{K}^{ij}h_{ij} - \xi\Delta h + \xi\nabla^i\nabla^jh_{ij} - \xi\bar{R}^{ij}h_{ij} = 0, \quad (3.2)$$

$$\nabla_i\kappa^{ij} - \lambda\nabla^j\kappa + \bar{K}^{kl}\nabla_k h_l^j - \frac{1+2\lambda}{2}\bar{K}^{kl}\nabla^jh_{kl} + \frac{1}{2}\bar{K}^{kj}\nabla_k h + \lambda\nabla^k\bar{K}h_k^j - \lambda\nabla^j\bar{K}^{kl}h_{kl} = 0, \quad (3.3)$$

$$\begin{aligned} & -\frac{\partial}{\partial t}(\kappa^{ij} - \lambda\bar{\gamma}^{ij}\kappa + \lambda\bar{K}h^{ij} - \lambda\bar{\gamma}^{ij}\bar{K}^{kl}h_{kl}) - n\left((1-2\lambda)\bar{K}\bar{K}^{ij} - \lambda\bar{K}^2\bar{\gamma}^{ij} + 2\bar{K}^{ik}\bar{K}_k^j\right) \\ & - (1-2\lambda)\bar{N}\bar{K}\kappa^{ij} - 2\bar{N}\bar{K}_l^i\kappa^{lj} - 2\bar{N}\bar{K}_l^j\kappa^{li} - \bar{N}\left((1-2\lambda)\bar{K}^{ij} - 2\lambda\bar{\gamma}^{ij}\bar{K}\right)\kappa \\ & - \lambda\bar{N}\bar{K}^2h^{ij} - 2\bar{N}\bar{K}^{ik}\bar{K}^{jl}h_{kl} - \bar{N}\left((1-2\lambda)\bar{K}^{ij} - 2\lambda\bar{\gamma}^{ij}\bar{K}\right)\bar{K}^{kl}h_{kl} \\ & + \xi\left[\nabla^i\nabla^jn - \bar{\gamma}^{ij}\Delta n - (\bar{R}^{ij} - \bar{\gamma}^{ij}\bar{R})n + \frac{\bar{N}}{2}\Delta h^{ij} - \frac{\bar{N}}{2}\nabla^k\nabla^ih_k^j - \frac{\bar{N}}{2}\nabla^k\nabla^jh_k^i\right. \\ & + \frac{\bar{N}}{2}\nabla^i\nabla^jh - \bar{N}\bar{\gamma}^{ij}\Delta h + \bar{N}\bar{\gamma}^{ij}\nabla^k\nabla^lh_{kl} - \frac{1}{2}\nabla_k\bar{N}\nabla^ih^{jk} - \frac{1}{2}\nabla_k\bar{N}\nabla^jh^{ik} \\ & + \frac{1}{2}\nabla_k\bar{N}\nabla^kh^{ij} + \bar{\gamma}^{ij}\nabla_k\bar{N}\nabla^lh^{lk} - \frac{1}{2}\bar{\gamma}^{ij}\nabla_k\bar{N}\nabla^kh \\ & \left. + \bar{N}\bar{R}_k^jh^{ik} + \bar{N}\bar{R}_k^ih^{jk} - \bar{N}\bar{R}^{kl}\bar{\gamma}^{ij}h_{kl} - \bar{N}\bar{R}h^{ij}\right] \quad (3.4) \\ & + \Delta\bar{N}h_{ij} + \bar{\gamma}^{ij}\nabla^k\nabla^l\bar{N}h_{kl} - \nabla^k\nabla^j\bar{N}h_k^i - \nabla^k\nabla^i\bar{N}h_k^j \Big] = 0, \end{aligned}$$

$$\dot{h}_{ij} = 2\bar{N}\kappa_{ij} + 2\bar{N}\bar{K}_j^k h_{ik} + 2\bar{N}\bar{K}_i^k h_{jk} + 2\bar{K}_{ij}n, \quad (3.5)$$

$$\begin{aligned} & 2\nabla_i\bar{K}\nabla^in + \Delta\bar{K}n + 2\nabla_i\bar{N}\nabla^i\kappa + \bar{N}\Delta\kappa + \bar{N}\bar{K}^{ij}\Delta h_{ij} \\ & + 2\bar{N}\nabla^k\bar{K}^{ij}\nabla_k h_{ij} - \bar{N}\nabla_j\bar{K}\nabla_i h^{ij} \\ & + \frac{\bar{N}}{2}\nabla^k\bar{K}\nabla_k h + 2\nabla^i\bar{N}\bar{K}^{kl}\nabla_i h_{kl} + \bar{N}\Delta\bar{K}^{ij}h_{ij} - \bar{N}\nabla^i\nabla^j\bar{K}h_{ij} \quad (3.6) \\ & - 2\nabla^i\bar{N}\nabla^j\bar{K}h_{ij} + 2\nabla^i\bar{N}\nabla_i\bar{K}^{kl}h_{kl} = 0. \end{aligned}$$

Here the indices are raised and lowered using the background metric  $\bar{\gamma}_{ij}$ , and the covariant derivatives are understood with respect to this metric.

One makes an important observation. As discussed above, eq. (3.6) is supposed to determine the evolution of the lapse. However, the terms linear in  $n$  disappear from this equation whenever the gradients of the background extrinsic curvature vanish. In particular, this happens for static or spatially homogeneous backgrounds. Then, instead of determining the lapse, eq. (3.6) imposes a constraint on an otherwise propagating field, making the extra mode non-dynamical.<sup>3</sup>

It is instructive to work out the case of Minkowski background in a certain detail. In

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<sup>3</sup> The fact that the extra mode does not propagate at linear level in homogeneous or static backgrounds also holds for the general Hořava action (2.1). The reason is that the combination in the square brackets in the secondary constraint (2.8) vanishes on these backgrounds. This is precisely the combination which multiplies the perturbation of the lapse in the linearized equation.

this case eq. (3.4) takes the form

$$-\frac{\partial}{\partial t}(\kappa_{ij} - \lambda\delta_{ij}\kappa) + \xi \left[ \partial_i\partial_j n - \delta_{ij}\Delta n + \frac{1}{2}\Delta h_{ij} - \frac{1}{2}\partial_k\partial_i h_{jk} - \frac{1}{2}\partial_k\partial_j h_{ik} + \frac{1}{2}\partial_i\partial_j h - \delta_{ij}\Delta h + \delta_{ij}\partial_k\partial_l h_{kl} \right] = 0. \quad (3.7)$$

The linearized Hamiltonian constraint (3.2) yields  $\Delta h = \partial_i\partial_j h_{ij}$ ; therefore the trace of (3.7) reads

$$(1 - 3\lambda)\dot{\kappa} = -2\xi\Delta n.$$

If  $n$  were determined by (3.6), this would be a first order equation for  $\kappa$ . However, in Minkowski (3.6) reduces to

$$\Delta\kappa = 0,$$

which restricts the perturbation of both the extrinsic curvature and the lapse to vanish. The rest of the argument proceeds in the same way as in GR and one concludes that all scalar modes are non-propagating.<sup>4</sup> The same effect occurs in any spatially homogeneous or static background. This explains why the extra mode was overlooked in the previous analyses [6–8] that focused on this class of backgrounds.

According to the above discussion the extra mode reveals itself only in backgrounds which are *both* time-dependent and spatially inhomogeneous. Finding an exact solution of Hořava gravity with these properties is a difficult task. Fortunately, for our purposes it is not needed: it is enough to realize that such backgrounds exist. As a concrete example, one can keep in mind a large gravitational wave.<sup>5</sup>

In the generic case, the system of linearized equations is intractable. What saves the day is the fact that the extra mode appears in *any* background such that the terms proportional to  $n$  in (3.6) do not cancel. It is enough to consider perturbations at space-time scales much shorter than the characteristic distance of the variation of the background. This allows to treat the background fields as almost constant at the scales of interest and ensures the validity of Fourier analysis at these scales. Technically this amounts to keeping in the equations only terms with least number of derivatives of the background. Let us make this point more quantitative. We assume that the background metric changes at characteristic space-time scale  $L$ . Then we have  $\bar{R}_{ij} \sim 1/L^2$ ,  $\bar{K}_{ij} \sim 1/L$ . We are interested in perturbations at distances much shorter than  $L$ . This means that we consider perturbations with frequencies and momenta  $\omega, p \gg 1/L$ . Consequently, in eqs. (3.2)–(3.6) we can neglect terms with derivatives of the background in comparison to the terms with derivatives of perturbations. This does not imply throwing away all the terms with background gradients: some of these terms may be the leading ones. For instance, the first term in eq. (3.6) is the leading contribution containing the lapse  $n$  in this equation. In this

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<sup>4</sup>This statement is true only at linear order in perturbations. The non-linear corrections will bring back the propagating mode as it is clear from the study of perturbations in general backgrounds and from the Stückelberg picture, see below.

<sup>5</sup>Moreover, once the perturbations to any metric are considered, the presence of inhomogeneities is universal.

way we obtain the simplified system,

$$-2\bar{K}_{ij}\kappa^{ij} + 2\lambda\bar{K}\kappa - \xi\Delta h + \xi\partial^i\partial^j h_{ij} = 0, \quad (3.8)$$

$$\partial_i\kappa^{ij} - \lambda\partial^j\kappa + \bar{K}^{kl}\partial_k h_l^j - \frac{1+2\lambda}{2}\bar{K}^{kl}\partial^j h_{kl} + \frac{1}{2}\bar{K}^{kj}\partial_k h = 0, \quad (3.9)$$

$$-\dot{\kappa}^{ij} + \lambda\delta^{ij}\dot{\kappa} - \lambda\bar{K}\dot{h}^{ij} + \lambda\delta^{ij}\bar{K}^{kl}\dot{h}_{kl} + \xi\left[\partial^i\partial^j n - \delta^{ij}\Delta n + \frac{\bar{N}}{2}\Delta h^{ij} - \frac{\bar{N}}{2}\partial^k\partial^i h_k^j - \frac{\bar{N}}{2}\partial^k\partial^j h_k^i + \frac{\bar{N}}{2}\partial^i\partial^j h - \bar{N}\delta^{ij}\Delta h + \bar{N}\delta^{ij}\partial^k\partial^l h_{kl}\right] = 0, \quad (3.10)$$

$$\dot{h}_{ij} = 2\bar{N}\kappa_{ij} + 2\bar{K}_{ij}n, \quad (3.11)$$

$$2\partial^i n \nabla_i \bar{K} + \bar{N}\Delta\kappa + \bar{N}\bar{K}^{ij}\Delta h_{ij} = 0, \quad (3.12)$$

where without loss of generality we have set<sup>6</sup>  $\bar{\gamma}_{ij} \approx \delta_{ij}$ . Let us first consider eq. (3.11). It follows from this equation that either  $\kappa_{ij}$  or  $\bar{K}_{ij}n$  is at least of order  $\omega h_{ij}$ . Let us further assume<sup>7</sup>  $\omega \gg p$ : we will see shortly that the dispersion relation of the extra mode obeys this inequality. Then we see that in eq. (3.12) the last term is always negligible. A similar reasoning shows that one can neglect all the terms containing  $h_{ij}$  in (3.10). This yields a closed system of equations for  $\kappa^{ij}$  and  $n$ :

$$-\dot{\kappa}^{ij} + \lambda\delta^{ij}\dot{\kappa} + \xi[\partial^i\partial^j n - \delta^{ij}\Delta n] = 0, \quad (3.13)$$

$$2\partial^i n \nabla_i \bar{K} + \bar{N}\Delta\kappa = 0. \quad (3.14)$$

We point out that this system is explicitly first order in time derivatives. As already mentioned, at short scales we can treat the combinations of the background fields appearing in the above equations as constant. One performs the Fourier decomposition

$$h_{ij}, \kappa^{ij}, n \propto e^{-i\omega t + i\mathbf{p}\mathbf{x}}$$

and finds that the solution of the system (3.13) – (3.14) has frequency

$$\omega = \frac{\xi\bar{N}p^4}{(1-3\lambda)p^j\partial_j\bar{K}}. \quad (3.15)$$

The extrinsic curvature for this solution is determined in terms of the lapse,

$$\kappa^{ij} = \frac{ip^k\partial_k\bar{K}}{\bar{N}p^4} \left( -(1-3\lambda)p^i p^j + (1-\lambda)\delta^{ij}p^2 \right) n. \quad (3.16)$$

Note that  $\omega \sim p^3 L^2$  which is indeed much bigger than  $p$ . Besides, the extrinsic curvature behaves as

$$\kappa^{ij} \sim \frac{n}{pL^2}. \quad (3.17)$$

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<sup>6</sup>The background metric can be always brought to this form in the vicinity of any given point by the time-independent 3-dimensional diffeomorphism.

<sup>7</sup>The analysis in the complementary regime  $\omega \sim p$  reveals only the two transverse traceless modes of the graviton.

From this estimate one concludes that the r.h.s. of (3.11) is dominated by the second term. This yields for the perturbations of the metric

$$h_{ij} = \frac{2i(1-3\lambda)p^k \partial_k \bar{K}}{\xi p^4} \bar{K}_{ij} n. \quad (3.18)$$

Finally, we have to check the constraints (3.8), (3.9). From (3.18) we see that

$$h_{ij} \sim \frac{n}{(pL)^3}.$$

This estimate together with (3.17) implies that the terms containing  $h_{ij}$  in (3.9) should be neglected compared to the first two terms; on the other hand, all the term in (3.8) are of the same order. Then it is straightforward to verify using the explicit expressions (3.16), (3.18) that the constraints are satisfied.

Let us briefly summarize our results. Eqs. (3.16) and (3.18) provide the explicit expression for the extra mode of Hořava gravity in the short wavelength limit. This mode is parametrized by a single scalar function  $n(\mathbf{p})$  and has the dispersion relation (3.15). Together with the two polarizations of the graviton found in the complementary regime  $\omega \sim p$  this matches with our counting of degrees of freedom in section 2. The frequency (3.15) of the mode diverges when the gradients of the background extrinsic curvature vanish. Thus the mode becomes singular for spatially homogeneous or static backgrounds.

The expression (3.15) also diverges for the modes with spatial momenta perpendicular to the gradient  $\partial_i \bar{K}$ . Naively, one could try to find the behaviour of these modes by applying the Fourier analysis to the system (3.8)–(3.12) including next to leading term from eq. (3.6), i.e. the term  $\Delta \bar{K} n$ . However, this would give an incorrect result. The leading-order expression (3.15) for the frequency of the mode depends on the background fields and hence on the space-time point where one performs the Fourier decomposition. Therefore at the subleading level the first term in eq. (3.6) produces contributions of the form

$$n t \partial_i \omega \partial^i \bar{K} \sim n \omega t L^{-3},$$

which at the time scales of interest  $t \sim \omega^{-1}$  are of the same order as the term

$$\Delta \bar{K} n \sim n L^{-3}.$$

Thus, the consistent treatment of the subleading effects requires going beyond Fourier analysis. Instead, one has to implement the WKB expansion in order to properly account for the inhomogeneity of the background. We will perform this study in the decoupling limit in section 5, where we will find that the subleading corrections generically lead to fast exponential instability of the extra mode.

To close this section, let us emphasize that the qualitative properties of the extra mode mentioned above are generic for any Hořava-type Lagrangian, even though in our consideration we mainly focused on the term  $(1-\lambda)K^2$ . Technically, the important point is that the perturbation of the lapse  $n$  enters in the linearization of the secondary constraint (2.8) with a coefficient that does not vanish on inhomogeneous and time-dependent backgrounds.

Then this equation relates  $n$  with the perturbation of the trace of the extrinsic curvature  $\kappa$  and leads to a first order evolution equation. In other words there will be always analogs of eqs. (3.13), (3.14) whenever the Lagrangian contains terms violating the four-dimensional general covariance. This is what makes the scalar mode propagate at linear level around inhomogeneous and time-dependent backgrounds for any Hořava-type Lagrangian.

#### 4 Stückelberg formalism

To get more insight into the dynamics of Hořava gravity we use the Stückelberg formalism. This will allow us to clearly separate the extra propagating mode and perform a detailed analysis of its properties.

The first step is to restore the full general covariance<sup>8</sup> at the expense of introducing the corresponding Stückelberg field. Namely, we encode the foliation structure of Hořava gravity in a scalar field  $\phi(x)$  with non-vanishing time-like gradient. The surfaces of the foliation are then defined by the equations

$$\phi(x) = const . \tag{4.1}$$

The original action (2.1) is written in the gauge where the field  $\phi$  coincides with time,  $\phi = t$ . Below we will refer to this choice of coordinates as “unitary gauge”.

Before obtaining the explicit expression for the action in an arbitrary gauge let us anticipate some of its properties. First, due to the presence of the new field  $\phi$ , we expect the action to be some kind of tensor-scalar theory. Second, the invariance of the original formulation (2.1) under time reparameterizations (second equation in (2.3)) translates into the symmetry of the covariant action with respect to reparameterizations of the Stückelberg field,

$$\phi \mapsto \tilde{\phi} = f(\phi) , \tag{4.2}$$

where  $f$  is an arbitrary monotonous function. The appearance of a time-dependent vev for  $\phi$  breaks the product of this symmetry and general covariance down to the diagonal subgroup. The latter translates in the unitary gauge into the invariance under foliation preserving diffeomorphisms (2.3).

To proceed one notices that the quantities appearing in (2.1) are the standard geometrical objects (induced metric, extrinsic and intrinsic curvature) characterizing the embedding of the hypersurfaces defined by (4.1) in space-time. The central object in the construction of these quantities is the unit normal vector<sup>9</sup>  $u_\mu$ . Explicitly,

$$u_\mu \equiv \frac{\partial_\mu \phi}{\sqrt{X}} ,$$

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<sup>8</sup>While this paper was in preparation ref. [12] appeared which also deals with the topic of the covariant form of the Hořava gravity.

<sup>9</sup>Throughout the paper the Greek indices  $\mu, \nu, \dots$  are raised and lowered using the 4-dimensional metric  $g_{\mu\nu}$  while the Latin indices  $i, j, \dots$  are raised and lowered using the spatial metric  $\gamma_{ij}$ . The same correspondence applies to the covariant derivatives carrying these indices.

where

$$X \equiv g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi .$$

Note that  $u_\mu$  is automatically invariant under the transformations (4.2). Other geometrical quantities associated to the foliation are constructed out of  $u_\mu$  and its derivatives. We have the following expressions for the spatial projector:

$$P_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu ,$$

the extrinsic curvature:

$$\mathcal{K}_{\mu\nu} \equiv P_{\rho\mu} \nabla^\rho u_\nu = \frac{1}{\sqrt{X}} P_\mu^\rho P_\nu^\sigma \nabla_\rho \nabla_\sigma \phi ,$$

and the intrinsic Riemann tensor:

$$\mathcal{R}^\mu_{\nu\rho\sigma} = P_\alpha^\mu P_\nu^\beta P_\rho^\gamma P_\sigma^\delta {}^{(4)}R^\alpha_{\beta\gamma\delta} + \mathcal{K}_\rho^\mu \mathcal{K}_{\nu\sigma} - \mathcal{K}_\sigma^\mu \mathcal{K}_{\nu\rho} , \quad (4.3)$$

where in the last equation  ${}^{(4)}R^\alpha_{\beta\gamma\delta}$  is the 4-dimensional Riemann tensor. Now it is straightforward to obtain the covariant form of the action (2.1) by identifying the quantities appearing in the ADM decomposition with the appropriate combinations of  $u_\mu$ ,  $P_{\mu\nu}$ ,  $\mathcal{K}_{\mu\nu}$ , etc. in the unitary gauge. For instance, in this gauge one has

$$\begin{aligned} u_0 &= \frac{1}{\sqrt{X}} = N , & u_i &= 0 , \\ P^{00} &= P^{0i} = 0 , & P^{ij} &= -\gamma^{ij} , \\ \mathcal{K}_{ij} &= K_{ij} , & \text{etc.} & \end{aligned} \quad (4.4)$$

In this way from (2.1) we obtain the following covariant action,

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{-g} \left\{ - {}^{(4)}R + (1 - \lambda) \mathcal{K}^2 + \zeta P^{\mu\nu} P^{\rho\sigma} \mathcal{R}_{\mu\rho} \mathcal{R}_{\nu\sigma} + \dots \right\} , \quad (4.5)$$

where  $\mathcal{K} \equiv \mathcal{K}_\mu^\mu$  and

$$\mathcal{R}_{\mu\rho} = P_\mu^\alpha P_\rho^\beta {}^{(4)}R_{\alpha\beta} - u^\alpha u^\beta {}^{(4)}R_{\alpha\mu\beta\rho} + \mathcal{K}_\alpha^\alpha \mathcal{K}_{\mu\rho} - \mathcal{K}_\mu^\alpha \mathcal{K}_{\alpha\rho} .$$

In deriving (4.5) we set for simplicity  $\xi = 1$ ; as we have already mentioned the value of this parameter is not physically relevant and we will stick to this choice from now on. The above action describes gravity interacting with a derivatively coupled scalar<sup>10</sup>  $\phi$ , which enters into (4.5) through the combinations  $P_{\mu\nu}$ ,  $\mathcal{K}_{\mu\nu}$ ,  $\mathcal{R}_{\mu\nu}$ .

From the action (4.5) the advantage of the Stückelberg formalism is clear: it allows to transfer the extra mode of Hořava gravity from the metric sector to the  $\phi$ -sector. Indeed, due to the general covariance of the action one can always choose the gauge where the metric sector contains only the two transverse traceless modes of the graviton. At the same time the extra mode is unambiguously identified with the fluctuation of the foliation structure.

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<sup>10</sup>Clearly, the terms in the action containing the scalar disappear when  $\lambda = 1$  and all the higher order terms vanish, i.e. in the pure GR case.

At this point we encounter a puzzle. To be consistent with the counting of degrees of freedom in the unitary gauge the equation of motion for the Stückelberg field must be first order in time derivatives. On the other hand, it is easy to see that the action (4.5) contains more than two derivatives of the field  $\phi$ . For example, consider the term proportional to  $(1 - \lambda)$ . Written explicitly in terms of  $\phi$  it reads,

$$S_\lambda = \frac{M_P^2(1 - \lambda)}{2} \int d^4x \sqrt{-g} \frac{1}{X} \left( \square\phi - \frac{\nabla^\mu\phi\nabla^\nu\phi}{X} \nabla_\mu\nabla_\nu\phi \right)^2, \quad (4.6)$$

and contains four derivatives.<sup>11</sup> Thus, for general choice of space-time coordinates, the equation of motion for  $\phi$  is fourth order in time derivatives. The corresponding Cauchy problem requires four arbitrary initial data; this apparently contradicts the counting of degrees of freedom performed in section 2, where we found only one additional function compared to GR. The resolution of this puzzle lies in the fact that the higher-derivative equation following from (4.6) is of a very special type. There exist a *particular* choice of coordinates for the formulation of the Cauchy problem where less initial data are required. This Cauchy slicing is precisely the preferred foliation of the model. In these coordinates the number of time derivatives in the equation for  $\phi$  is reduced to one, which matches with the counting of the degrees of freedom in the unitary gauge.

To illustrate the point about the reduction of degrees of freedom in a preferred frame let us make a digression and consider the following equation for a non-relativistic scalar  $\varphi$ ,

$$\ddot{\varphi} + (-1)^q \Delta^q \varphi = 0, \quad (4.7)$$

where  $q \geq 2$  is an integer number. This equations describes one degree of freedom and the corresponding Cauchy problem involves two initial data. The general solution of the equation (4.7) is a collection of waves with dispersion relation  $\omega^2 = p^{2q}$ .

However, when one attempts to write down the same theory in a manifestly Lorentz invariant way one seemingly encounters the problem that the theory is higher derivative both in space and time directions. Indeed, in a generic Lorentz frame eq. (4.7) acquires up to  $2q$  time derivatives of  $\varphi$ . An observer in this frame would conclude that the number of degrees of freedom is  $q > 1$ , as to solve the equations of motion she would need up to  $2q$  initial conditions. Does this contradict the counting of degrees of freedom in the original frame? The answer is no, because the two formulations of the Cauchy problem are *physically inequivalent*. Indeed, recall that specification of the solutions which are considered as physical involve fixing the boundary conditions at spatial infinity. In the simple case of eq. (4.7) the natural choice is to impose vanishing of the field,

$$\varphi \rightarrow 0, \quad \text{at } |\mathbf{x}| \rightarrow \infty. \quad (4.8)$$

When one thinks of the Cauchy problem in the boosted frame, one also implicitly assumes the condition (4.8) but now imposed at the spatial infinity in this frame. Then the standard

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<sup>11</sup>The terms with gradients of the intrinsic curvature contribute with even higher derivatives of  $\phi$ .

procedure is to perform the Fourier expansion of the initial data and follow the evolution of each eigenmode separately. In this approach one indeed finds that in the boosted frame there are  $2q$  eigenmodes. However, a straightforward analysis shows that only two of these modes have real frequencies. This means that the other modes grow exponentially either at positive or at negative times. In particular, if an observer living in the boosted frame is free to choose arbitrary initial conditions she will conclude that the system is unstable. However, the growing modes, while being legitimate solutions in the boosted frame, do not satisfy the condition (4.8) in the original coordinate system. Thus in the latter system they are discarded as unphysical.

The situation here is similar to the case of fields obeying second-order equation of motion

$$\ddot{\varphi} - v^2 \Delta \varphi = 0 \tag{4.9}$$

with superluminal velocity,  $v > 1$ . The Cauchy problems for eq. (4.9) are physically inequivalent when formulated in the original and highly boosted frames [13–15]. The latter corresponds to the case when the Cauchy slices intersect the future causal cone

$$t = |\mathbf{x}|/v \tag{4.10}$$

of eq. (4.9). On the other hand, all the Cauchy problems formulated on slices lying outside the cone (4.10) are equivalent. The difference between eqs. (4.7) and (4.9) is that the former does not have a well-defined causal cone: the signal can propagate from the origin to any point at  $t > 0$ . Thus even arbitrarily small deviations from the original slicing qualitatively change the properties of the system.

For what follows it is convenient to slightly generalize the above discussion. Consider the Lorentz covariant equation

$$A^{\mu\nu} \partial_\mu \partial_\nu \varphi - (B^{\mu\nu} \partial_\mu \partial_\nu)^q \varphi = 0, \tag{4.11}$$

where the symmetric matrices  $A^{\mu\nu}$ ,  $B^{\mu\nu}$  transform as tensors under Lorentz boosts. These matrices may depend on various fields present in the theory, in particular, they can depend on the field  $\varphi$  itself. Naively, eq. (4.7) contains  $2q$  time derivatives and thus describes  $q$  degrees of freedom. However this reasoning is not correct in general. The properties of the differential equation (4.11) are characterized by the eigenvalues and eigenvectors of  $A^{\mu\nu}$  and  $B^{\mu\nu}$ . The general study of the possible cases is beyond the scope of the present paper. Here we just point out the special case when  $B^{\mu\nu}$  has one hypersurface-orthogonal timelike eigenvector with zero eigenvalue and 3 spacelike eigenvectors<sup>12</sup> with non-zero eigenvalues. Then, in the frame defined by these eigenvectors the number of time derivatives in the operator  $B^{\mu\nu} \partial_\mu \partial_\nu$  is reduced. In general, in curved backgrounds or when the matrix  $B^{\mu\nu}$  is space-time dependent, the resulting operator still contains one time derivative, cf. eq. (4.12) below. Still, the main conclusion is that the number of degrees of freedom described by an equation can be reduced compared to the naive expectation by proper choice of Cauchy slicing.

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<sup>12</sup>Note that because of symmetry of the matrix  $B^{\mu\nu}$  its eigenvectors are orthogonal.



Let us return to the Hořava gravity. Consider the higher derivative terms appearing in the equation of motion for the Stückelberg field  $\phi$ . To be concrete let us take the case of the action (4.6). Then the term with four derivatives in the equation reads<sup>13</sup>

$$(P^{\mu\nu}\nabla_\mu\nabla_\nu)^2\phi .$$

This is precisely of the form (4.11) with  $B^{\mu\nu} = P^{\mu\nu}$ ,  $q = 2$ . Thus we expect a reduction of the number of degrees of freedom in a certain frame. In the rest of this section we demonstrate that this is indeed the case for a large class of Hořava-type Lagrangians. Namely, we prove the following statement. Consider perturbations of the field  $\phi$  around the background  $\bar{\phi}$ ,

$$\phi = \bar{\phi} + \chi . \tag{4.13}$$

Then in the frame where the background is in unitary gauge,

$$\bar{\phi} = t , \tag{4.14}$$

the linearized equation for  $\chi$  is first order in time derivative.

For simplicity, we concentrate on the case when the Lagrangian for the Stückelberg field depends only on first derivatives of the normal vector  $u^\mu$ . Moreover, we assume that these derivatives enter into the Lagrangian through the extrinsic curvature  $\mathcal{K}_{\mu\nu}$ ,

$$\mathcal{L} = \mathcal{L}(u^\mu, \mathcal{K}_{\mu\nu}) . \tag{4.15}$$

This case covers all terms in the general Hořava-type Lagrangian except those involving spatial derivatives of the 3-dimensional curvature tensor  $R_{ijkl}$ . Indeed, the Gauss-Codazzi equation (4.3) implies that the terms polynomial in  $R_{ijkl}$  depend on  $\phi$  only through the projector  $P^{\mu\nu}$  and the extrinsic curvature.

One observes that the equation of motion for the field  $\phi$  has the form of the current conservation,

$$\nabla_\mu J^\mu = 0 , \tag{4.16}$$

where

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi} - \nabla_\nu \frac{\partial\mathcal{L}}{\partial\nabla_\mu\nabla_\nu\phi}$$

is the current related to the reparameterization symmetry (4.2). Let us demonstrate that the current is orthogonal to the gradient of  $\phi$ ,

$$u_\mu J^\mu = 0 . \tag{4.17}$$

By explicit computation we find,

$$J^\mu = \frac{1}{\sqrt{X}} \left\{ P_\sigma^\mu \frac{\partial\mathcal{L}}{\partial u_\sigma} - u_\sigma \mathcal{K}_\rho^\mu \frac{\partial\mathcal{L}}{\partial\mathcal{K}_{\sigma\rho}} + P_\sigma^\mu u_\rho \mathcal{K} \frac{\partial\mathcal{L}}{\partial\mathcal{K}_{\sigma\rho}} - P_\sigma^\mu P_\rho^\lambda \nabla_\lambda \frac{\partial\mathcal{L}}{\partial\mathcal{K}_{\sigma\rho}} \right\} . \tag{4.18}$$

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<sup>13</sup> Explicitly in terms of 3+1 decomposition we have the operator identity

$$P^{\mu\nu}\nabla_\mu\nabla_\nu = -\Delta + \mathcal{K}u^\lambda\nabla_\lambda \tag{4.12}$$

valid for the action on scalar functions.

This expression explicitly satisfies (4.17). Now we perform the separation of the field into the background and perturbations. The key observation is that in the frame defined by eq. (4.14) the current contains exactly one time derivative of the linear perturbation  $\chi$ . Indeed, the perturbations of  $u^\mu$  and the extrinsic curvature do not contain time derivatives of  $\chi$ . This follows from the explicit expressions

$$\begin{aligned} \delta u^\mu &= \frac{1}{\sqrt{\bar{X}}} \bar{P}^{\mu\nu} \partial_\nu \chi, \\ \delta \mathcal{K}_{\mu\nu} &= \frac{1}{\sqrt{\bar{X}}} \left[ -2\bar{a}_{(\nu} \bar{P}_{\mu)}^\rho \partial_\rho \chi - \bar{u}_\mu \bar{\mathcal{K}}_\nu^\rho \partial_\rho \chi + \bar{P}_\mu^\lambda \nabla_\lambda (\bar{P}_\nu^\rho \partial_\rho \chi) \right], \end{aligned}$$

where bar refers to the background values and

$$a^\nu \equiv u^\lambda \nabla_\lambda u^\nu$$

is the proper acceleration of the congruency defined by  $u^\mu$ . Then, by inspection of the expression (4.18) one finds that the only contribution with time derivative comes from the variation of the factor  $1/\sqrt{\bar{X}}$ ,

$$\delta \frac{1}{\sqrt{\bar{X}}} = -\frac{1}{\bar{X}} \bar{u}^\sigma \partial_\sigma \chi = -\frac{1}{\sqrt{\bar{X}}} \dot{\chi} + \dots$$

To complete the argument we have to show that taking divergence of the current in the equation of motion (4.16) does not bring more time derivatives. Due to the property (4.17) one has

$$J^\mu = P_\nu^\mu J^\nu,$$

and thus (4.16) can be written as

$$P_\nu^\mu \nabla_\mu J^\nu - a_\nu J^\nu = 0.$$

When expanded to linear order in perturbations, the first term contains only spatial derivatives of the perturbation of the current. Thus this term remains first derivative in time. Explicitly, the corresponding contribution reads,

$$-\frac{\bar{u}^\mu \bar{J}^\nu}{\sqrt{\bar{X}}} \nabla_\mu \nabla_\nu \chi.$$

Another contribution with first time derivatives comes from the perturbation of  $a_\nu$ ,

$$\delta a^\nu = \frac{\bar{u}^\lambda \bar{P}^{\nu\rho}}{\sqrt{\bar{X}}} \nabla_\lambda \nabla_\rho \chi + \dots$$

Note that the two contributions are equal and sum up. This completes the proof.

Two comments are in order. First, let us take a closer look at the equation of motion for the Stückelberg field. For the sake of the argument let us consider the action (4.6). Up to an irrelevant constant factor the current (4.18) reduces in this case to

$$J^\mu = -\frac{1}{\sqrt{\bar{X}}} P^{\mu\nu} \nabla_\nu \mathcal{K}.$$

In the unitary gauge it takes the form<sup>14</sup>

$$J^0 = 0, \quad J^i = N^2 \nabla^i K .$$

One observes that the equation of motion (4.16) coincides with the secondary constraint (3.1). It is straightforward to check that the equation for  $\phi$  is identical to the secondary constraint also for the general Hořava type Lagrangian. This confirms the consistency of the Stückelberg treatment.

Second, the expression (4.15) is not the most general Lagrangian compatible with the reparameterization symmetry (4.2). Even if one restricts attention to Lagrangians with only first derivatives of  $u^\mu$ , one can still add to (4.15) dependence on  $a^\nu$ . In the unitary gauge the terms with  $a^\nu$  translate into terms with spatial derivatives of the lapse,

$$a^0 = 0, \quad a^i = \nabla^i N / N ;$$

such terms were not considered in [1]. As the linear expansion of  $a^\nu$  contains one time derivative of the perturbation  $\chi$ , the argument given above does not go through in this case, and the equation for the Stückelberg field may contain more than one time derivative.<sup>15</sup> In particular, adding the term  $(a^\nu)^2$  to the Lagrangian makes the equation second order in time.

## 5 Instability and strong coupling

We now perform a detailed study of the properties of the Stückelberg field perturbations  $\chi$ . To make the analysis clear we restrict the study to the case when the Stückelberg action contains only the term (4.6). Moreover, we consider the decoupling limit,  $(1 - \lambda) \ll 1$ . In this limit the backreaction of the Stückelberg sector on the space-time geometry is negligible and one considers  $\phi$  as propagating in a fixed background metric. At the same time this is precisely the limit where the theory is expected to approach GR. In this regime the non-linearities of the Stückelberg field become important at energy scales much smaller than those of the other modes in the theory. This will allow to easily establish the strong coupling scale in this sector.

We start by obtaining the quadratic action for the perturbations. According to the discussion of the previous section we work in the foliation defined by the background value of the field. Thus we fix the gauge  $\bar{\phi} = t$ ,  $\bar{N}_i = 0$ . Expanding the quantities in the action (4.6) to quadratic order one obtains

$$S_\chi = \frac{M_P^2(1 - \lambda)}{2} \int d^4x \sqrt{\gamma} \left[ -2\bar{N}^2 \nabla_i \bar{K} \dot{\chi} \nabla^i \chi + \bar{N}^3 (\Delta \chi)^2 - 2\bar{N} \nabla_i \nabla_j \bar{N}^2 \nabla^i \chi \nabla^j \chi + \left( \frac{2}{3} \Delta \bar{N}^3 - \bar{N}^2 \dot{\bar{K}} \right) \nabla_i \chi \nabla^i \chi \right] . \tag{5.1}$$

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<sup>14</sup>Recall that the covariant derivatives with Latin indices refer to the 3-dimensional metric  $\gamma_{ij}$ .

<sup>15</sup> This is readily understood in the unitary gauge, where the secondary constraint will now contain the time derivative of the lapse.

where we have used

$$\begin{aligned} \mathcal{K} = & \bar{K} - 2\nabla_i \bar{N} \nabla^i \chi - \bar{N} \Delta \chi + \bar{N} \dot{\chi} \Delta \chi + 2\bar{N} \nabla^i \dot{\chi} \nabla_i \chi \\ & + 2\nabla_i \bar{N} \dot{\chi} \nabla^i \chi + \dot{\bar{N}} \nabla_i \chi \nabla^i \chi + \frac{\bar{N}^2}{2} \bar{K} \nabla_i \chi \nabla^i \chi - \bar{N}^2 \bar{K}^{ij} \nabla_i \chi \nabla_j \chi . \end{aligned}$$

This action explicitly reveals the properties of the extra mode discussed previously. Indeed, it contains only one time derivative of  $\chi$ , so that the resulting equation is first order in time. The term with time derivative vanishes whenever the gradient of the extrinsic curvature is zero, and the field  $\chi$  becomes non-propagating. One also observes that due to the background equation of motion (3.1) this action is invariant under the shifts

$$\chi \mapsto \chi + \xi(t) ,$$

where  $\xi(t)$  is an arbitrary function of time. This is recognized as the linearized form of the reparameterization (4.2). This symmetry prevents  $\chi$  from having the ordinary  $\dot{\chi}^2$  kinetic term.

Let us analyze the equation of motion following from (5.1). As in section 3 we are interested in the short wavelength limit,

$$pL \gg 1 ,$$

where  $p$  is the momentum of the  $\chi$ -mode, and  $L$  is the characteristic length of the variation of the background. In this regime the dominant terms in the equation are those containing the largest number of derivatives of  $\chi$ . After using the equations for the background (3.1), the leading contributions to the equations of motion are

$$2\nabla^i \bar{K} \partial_i \dot{\chi} + \bar{N} \Delta^2 \chi + 6\nabla^i \bar{N} \nabla_i \Delta \chi = 0 , \quad (5.2)$$

where for future reference we retain the main subleading correction represented by the last term on the l.h.s. To find the solution of eq. (5.2) we use the same strategy as in section 3. Namely, we restrict to the vicinity of a given point  $x_o$ . To the leading approximation the background fields in this vicinity can be considered as constant. One also assumes that the spatial metric  $\bar{\gamma}_{ij}$  at this point is flat and the Cristoffel symbols vanish; this can be always achieved by performing a 3-dimensional diffeomorphism. Now it is easy to obtain the leading behavior of the solution. One performs the Fourier expansion

$$\chi \propto e^{-i\omega t + i\mathbf{p}\mathbf{x}} \quad (5.3)$$

and substitutes it into (5.2). Discarding the last — subleading — term one obtains the dispersion relation

$$\omega = -\frac{\bar{N} p^4}{2(p^i \partial_i \bar{K})} . \quad (5.4)$$

This coincides with the expression<sup>16</sup> (3.15) of section 3 in the decoupling limit  $\lambda \approx 1$ .

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<sup>16</sup>Recall that in the present section we work within the convention  $\xi = 1$ .

We now discuss the first corrections in  $1/(pL)$  to the solution (5.3), (5.4). The reason for considering these corrections is that they qualitatively change the behaviour of the mode making it either exponentially decaying or growing. As already pointed out in section 3, in order to find the subleading corrections one has to go beyond the Fourier analysis and implement the WKB expansion in the vicinity of the point  $x_o$ . The details of this procedure are contained in the appendix. Here we quote the main result. The eigenmode frequency acquires an imaginary part which is estimated as

$$\delta\omega \sim ip^2 L . \tag{5.5}$$

The sign of this imaginary part depends on the direction of the mode momentum  $\mathbf{p}$  relative to the gradients of the background. Those modes for which the imaginary part is positive are exponentially growing. Note that the rate of this growth is much faster than the characteristic background frequency  $1/L$ . Thus we conclude that the Hořava model suffers from fast instability at short scales.

Another problem with the theory appears when one takes into account self-interaction of the field  $\chi$ . Let us first consider the case of flat background. Then, computing the leading non-linear terms from the expansion of the covariant Lagrangian (4.6) in the perturbation  $\chi$  we obtain,

$$S = \frac{M_P^2(1-\lambda)}{2} \int d^4x \{(\Delta\chi)^2 + 2\dot{\chi}((\Delta\chi)^2 + 2\partial_i\chi\partial_i\Delta\chi)\} . \tag{5.6}$$

As expected, the quadratic part of the action does not contain time derivatives. Note, however, that the time derivatives do appear in the interaction. The form of this Lagrangian is restricted by the fact that  $\chi$  nonlinearly realizes the field-reparameterization symmetry (4.2)

$$\chi \mapsto \chi + \xi(t) + \dot{\xi}(t)\chi + \dots .$$

The theory (5.6) clearly has a dimensionful coupling constant  $(\sqrt{|1-\lambda|} M_P)^{-1}$  which signals the presence of strong quantum coupling at high enough energies. The naive estimate of the strong coupling scale is provided by the inverse of this coupling [6],

$$\Lambda_{\text{naive}} = \sqrt{|1-\lambda|} M_P . \tag{5.7}$$

Note that this scale goes to zero in the putative GR limit  $\lambda \rightarrow 1$ . In particular, in this limit it is parametrically smaller than the “deep UV” scale  $M_P$  at which the higher-order terms of the Hořava model could become important.

We now argue that the scale of strong coupling for the action (5.6) is even lower than (5.7), viz. zero. The physical reason is that due to the absence of time derivatives in the quadratic part of the action, rapid fluctuations of the field  $\chi$  are not suppressed. Hence, the interaction terms with time derivatives blow up (see related discussion in [16]).

To make a quantitative statement let us regulate the action (5.6) by expanding on a nearly flat but nontrivial background. This introduces the first order kinetic term for  $\chi$  as in (5.1). For momenta much larger than the scale defined by the background the leading

order terms in the action are essentially the same as in flat space plus the background dependent kinetic term. Schematically,

$$S = \Lambda_{\text{naive}}^2 \int d^4x \{ L^{-2} v^i \dot{\chi} \partial_i \chi + (\Delta \chi)^2 + \dot{\chi} (\Delta \chi)^2 \} ,$$

where  $v^i$  is the unit vector along the direction of the extrinsic curvature gradient, and  $L$  is the typical length scale of the background. The free part of the action is invariant under the scaling<sup>17</sup>

$$\begin{aligned} x &\mapsto b^{-1} x \\ t &\mapsto b^{-3} t \\ \chi &\mapsto b \chi . \end{aligned}$$

Under this scaling the interaction term has dimension +4; thus it becomes relevant at short scales. To estimate the cutoff one performs the rescaling

$$\begin{aligned} t &\mapsto \hat{t} = t L^2 , \\ \chi &\mapsto \hat{\chi} = L^{-1} \Lambda_{\text{naive}} \chi , \end{aligned}$$

which brings the quadratic part of the action to the canonically normalized form. This yields,

$$S = \int d^4x \left\{ v^i \dot{\hat{\chi}} \partial_i \hat{\chi} + (\Delta \hat{\chi})^2 + \frac{L^3}{\Lambda_{\text{naive}}} \dot{\hat{\chi}} (\Delta \hat{\chi})^2 \right\} ,$$

where dot is now understood as the derivative with respect to the rescaled time  $\hat{t}$ . From this expression one reads out the cutoff scales for the spatial momentum and the rescaled frequency: they are set by the appropriate powers of the unique coupling constant appearing in the interaction term. One obtains

$$\begin{aligned} \Lambda_p &= L^{-3/4} \Lambda_{\text{naive}}^{1/4} , \\ \Lambda_{\hat{\omega}} &= L^{-9/4} \Lambda_{\text{naive}}^{3/4} . \end{aligned}$$

Using the relation  $\hat{\omega} = \omega/L^2$  between the rescaled and the physical frequencies we obtain the physical frequency cutoff

$$\Lambda_\omega = L^{-1/4} \Lambda_{\text{naive}}^{3/4} . \tag{5.8}$$

Note that due to the non-relativistic structure of the theory the cutoff scales in spatial momentum and frequency are not equal. Instead, they satisfy the relation  $\Lambda_\omega = \Lambda_p^3 L^2$ , which is compatible with the dispersion relation of the extra mode. It is clear that both  $\Lambda_p$  and  $\Lambda_\omega$  go to zero in the limit of flat background. Hence, the theory becomes strongly coupled at all scales.

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<sup>17</sup>Remarkably, we find here the same relative scaling of space and time as that proposed by Hořava for the deep UV to make the theory naively power-counting renormalizable. We have nothing more to say on that coincidence.

## 6 Hořava gravity with projectable lapse as a ghost condensate

In this section we consider another version of Hořava theory also proposed in [1]. In this formulation the lapse function appearing in the action (2.1) is assumed to be projectable, which means that it does not depend on spatial coordinates,  $N = N(t)$ . As we are about to see, the dynamics of the model in this case is very different from the case studied in the main body of the paper. We discuss it here only briefly, leaving a more thorough study for the future.

In the projectable case the variation of the action with respect to  $N$ , instead of producing the local Hamiltonian constraint (2.4), gives the integral of eq. (2.4) over the whole space. Such an integral constraint does not affect local physics. Thus, as far as local dynamics is concerned, the full set of equations of motion is provided<sup>18</sup> by eqs. (2.5)–(2.7). Using the reparameterization of time one can set  $N = 1$  in these equations. Let us count the number of required Cauchy data. Out of 12 functions present in  $\gamma_{ij}$ ,  $K^{ij}$ , 3 are constrained by eq. (2.5). 3 more functions are removed by residual spatial diffeomorphisms. Thus we are left with  $12 - 3 - 3 = 6$  initial conditions. This is larger by 2 than in the GR case, suggesting the presence of an extra mode with second order evolution equation.

The simplest way to study this new mode is by using the Stückelberg formalism. First, notice that the constraint  $N = 1$  can be enforced by adding to the action (2.1) the term

$$S_\rho = \int d^4x \sqrt{\gamma} N \frac{\rho}{2} \left( \frac{1}{N^2} - 1 \right),$$

where  $\rho$  is a Lagrange multiplier. From the relation (4.4), the generally covariant form of the previous action reads

$$S_\rho = \int d^4x \sqrt{-g} \frac{\rho}{2} (\nabla_\mu \phi \nabla^\mu \phi - 1). \tag{6.1}$$

Apart from this term the action for  $\phi$  contains higher-order contributions coming from the rest of the Hořava Lagrangian. The latter contributions either contain more than two derivatives or describe non-minimal couplings to the metric, cf. eq. (4.5).

The theory (6.1), together with higher order contributions, can be viewed as a special (non-minimally coupled) case of the ghost condensate model [10]. The latter model is characterized by a Lagrangian

$$\mathcal{L}_{gc} = P(\nabla_\mu \phi \nabla^\mu \phi) + (\text{terms with higher derivatives}), \tag{6.2}$$

where the function  $P(X)$  has a minimum at  $X = 1$ . This forces the field  $\phi$  to develop a non-zero time-like gradient. The Lagrangian (6.1) is the “sigma model limit” of (6.2) when the function  $P$  gets replaced by the constraint. Note, however, an important difference in the interpretation of the higher derivative terms in the ghost condensate model and Hořava gravity. In the standard approach to the ghost condensate they are considered as higher order terms in the effective field theory expansion, while according to Hořava they should be taken at face value and determine the UV properties of the system.

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<sup>18</sup>Clearly, eq. (2.8) expressing the conservation of the Hamiltonian constraint is also absent.

Let us get more insight into the dynamics of the field  $\phi$  described by (6.1). For the sake of the argument we omit the higher order terms. This would correspond to the IR limit of the putative UV complete theory. However, we will see shortly that the  $\phi$ -sector exhibits a number of pathologies which make the UV completion problematic.

The action yields the following equations of motion,

$$\begin{aligned}\nabla_\mu \phi \nabla^\mu \phi &= 1, \\ \nabla_\mu (\rho \nabla^\mu \phi) &= 0.\end{aligned}\tag{6.3}$$

These equations are equivalent to the equations of motion of an ideal pressureless fluid (dust) with energy-momentum tensor

$$T_{\mu\nu} = \rho \nabla_\mu \phi \nabla_\nu \phi.$$

We see that  $\rho$  and  $\nabla_\mu \phi$  are identified respectively as the energy density and velocity of the effective fluid. This result is in agreement with the findings of [11] where it was proposed to interpret the effective dust-like component arising in the projectable version of the Hořava gravity as dark matter. However, we point out that a well-known property of pressureless fluids is that they develop caustics, i.e. there are space-time regions where the fluid velocity is ill-defined (see [17] for discussion of this topic in the context of field theory models of dark matter). The formation of caustics is easy to understand in the decoupling limit when the backreaction of the fluid on the geometry is negligible. Then the fluid particles move along geodesics without feeling each other. Given a general inhomogeneous initial distribution of particle velocities, their trajectories will cross forming a caustic. While this process does not pose a problem for real dust where it leads to virialization, the formation of caustics means an inconsistency in the case of a scalar theory, as the field is not differentiable at the caustics. Thus, we conclude that for generic initial configurations the theory described by the action (6.1) breaks down after finite time evolution. Note that the formation of caustics is a general problem of the ghost condensate action (6.2) [18]. In that context, it was suggested [18] that the problem might be resolved by the effect of higher derivative term which make the fluid particles deviate from the geodesic motion. This hope is absent in the special case of the action (6.1): irrespective of the higher-order terms the fluid particles move exactly along geodesics as long as the constraint (6.3) is present.

The consistency of the Lagrangian (6.1) is also challenged at the quantum level. To get the flavor of the problem let us again omit the higher order terms and proceed to quantize the model in the canonical formalism. For simplicity we also neglect the backreaction of the field  $\phi$  on the metric and assume the latter to be flat. The canonically conjugate momenta for the variables  $\phi, \rho$  are

$$\begin{aligned}\pi_\phi &= \rho \dot{\phi}, \\ \pi_\rho &= 0.\end{aligned}\tag{6.4}$$

The constraint (6.3) in canonical variables takes the form

$$\frac{\pi_\phi^2}{\rho^2} - (\partial_i \phi)^2 = 1.\tag{6.5}$$



Equations (6.4) and (6.3) form a pair of second class constraints and enable to eliminate the variables  $\rho, \pi_\rho$ . In this way one obtains the Hamiltonian,

$$\mathcal{H} = \pi_\phi \sqrt{1 + (\partial_i \phi)^2} . \tag{6.6}$$

Note that in our case the elimination of constraints does not modify the Poisson brackets of the remaining variables,

$$\{\pi_\phi(\mathbf{x}), \phi(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \text{ etc.}$$

The quantization of the theory proceeds now in the standard way by imposing the canonical commutation relations on  $\pi_\phi$  and  $\phi$ .

The Hamiltonian (6.6) certainly looks unusual. To get insight into its properties, let us expand it around the background  $\phi = t, \pi_\phi = \rho_0$ . Thus we write

$$\phi = t + \chi/\sqrt{\rho_0}, \quad \pi_\phi = \rho_0 + \pi_\chi \sqrt{\rho_0}$$

and obtain,<sup>19</sup>

$$\mathcal{H} = \frac{1}{2}(\partial_i \chi)^2 + \frac{1}{2\sqrt{\rho_0}} \pi_\chi (\partial_i \chi)^2 + \dots . \tag{6.7}$$

As an example we included here one of the interaction terms. Now, (6.7) describes a theory with dimensionful coupling  $1/\sqrt{\rho_0}$ . This implies that the theory gets strongly coupled at the cutoff scale  $\Lambda \lesssim \rho_0^{1/4}$ . For the dark matter interpretation of the  $\phi$ -sector, taking the present-day value for the average density of dark matter, one obtains the cutoff

$$\Lambda \lesssim 10^{-3} \text{eV} ,$$

which is unacceptably low for a candidate theory of quantum gravity. In fact, the cutoff in the theory (6.7) may be even lower, viz. zero. This can be argued from the fact that the quadratic part of the Hamiltonian (6.7) does not contain the momentum  $\pi_\chi$ . Thus the quantum fluctuations of  $\pi_\chi$  are not suppressed and the interaction terms containing  $\pi_\chi$  blow up. The careful analysis of this issue is beyond the scope of this paper.<sup>20</sup>

To summarize, we found that in the decoupling limit, when the backreaction of the extra mode on the metric can be neglected, the projectable version of the Hořava gravity suffers from caustic and low cutoff problems. This suggests the inconsistency of the model in its present form.

## 7 Conclusions and discussion

In this paper we have studied Hořava’s proposal for quantum gravity [1] from the low-energy perspective. We have mainly concentrated on the “non-projectable” version of the model. We have uncovered the additional scalar degree of freedom arising from the explicit

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<sup>19</sup>Note that the canonical transformation from the variables  $\phi, \pi_\phi$  to  $\chi, \pi_\chi$  is time-dependent. Taking this properly into account eliminates the term linear in  $\pi_\chi$  in the Hamiltonian.

<sup>20</sup>Another issue which we do not address in this paper is the effect of the higher order terms on the power-counting.

breaking of the general covariance and analyzed its properties in detail. A peculiarity of the new mode is that it satisfies an equation of motion that is of first order in time derivatives; this means that it adds just one direction to the phase space, or ‘half’ a degree of freedom. At linear level the mode is manifest only on spatially inhomogeneous *and* time-dependent backgrounds. We have demonstrated the existence of the extra mode in two physically equivalent ways. First we performed the analysis in the original ADM-like formulation of Hořava and identified the mode among the perturbations of the metric. Second, we made use of the Stückelberg formalism which amounted to restoring the general covariance of the model. This required introducing a scalar field describing the foliation structure of the Hořava model. It is worth stressing that this procedure did not add any new physical degrees of freedom compared to the original formulation because the scalar field was introduced simultaneously with a new local symmetry. The latter gauge symmetry is the part of general covariance which is explicitly broken in the Hořava’s original formulation. The Stückelberg approach allowed us to transfer the extra mode into the fluctuations of the foliation structure and to perform a detailed study of its properties.

We found two serious problems associated with this mode. First, the mode develops very fast exponential instabilities at short distances. Second, it becomes strongly coupled at extremely low cutoff scale. Due to the non-relativistic nature of the theory the cutoff scales in spatial momentum and frequency are different. They both depend on the curvature of the background metric and go to zero when this curvature vanishes. These features allow to conclude that Hořava’s proposal is inconsistent in the present form.

We have also discussed the “projectable” version of the Hořava model. We have argued that this version can be understood as a certain limit of ghost condensation. In this case, the theory propagates a whole degree of freedom with second order equation in time for any background. In this sense the projectable version is better behaved than the non-projectable one. However, the preliminary analysis of the model in the regime when the backreaction of the extra mode on the space-time geometry can be neglected suggests that the model is still problematic since the additional field generically forms caustics and, again, has very low strong coupling scale.

Let us comment on the possible ways in which the problems of the theory can be addressed. The comparison of projectable and non-projectable cases suggests that the only way to make the extra mode well-behaved is to promote it to a full-fledged scalar. There are two strategies to do so. One possibility is to relax the requirement of invariance under reparameterizations of time and reduce the symmetry group of the theory down to (time-dependent) spatial diffeomorphisms. As a result one obtains some version of the ghost condensate model [10] which is known to be a consistent effective theory up to relatively high scales. The problem with caustics can be also addressed in this framework [18]. In the unitary gauge the theory would be described by a generalization of Hořava action containing arbitrary functions of the lapse  $N$ . Needless to say, it is unclear if the appealing UV properties of the Hořava proposal can be preserved within this approach. An argument suggesting that it may be impossible is the break-down of the black hole thermodynamics in the ghost condensate model [19, 20], and more generally, in theories with low-energy violation of Lorentz invariance [21]. Indeed, the break-down of thermodynamics indicates

violation of unitarity in the underlying theory, which makes construction of a renormalizable and hence UV complete quantum theory having the ghost condensate as its low-energy limit problematic.

Alternatively, one can promote the extra mode to a whole scalar without breaking the invariance under foliation-preserving diffeomorphisms. To achieve this one has to add to Hořava's Lagrangian terms that are non-linear in  $N$  and respect the symmetry. An example of such term is  $N^{-2}(\nabla_i N)^2$ . In the Stückelberg picture around flat background this corresponds to the addition of the term  $(\partial_t \chi)^2$ . In this case also the equation for the Stückelberg field  $\chi$  becomes of second order in time for any background. Still, the terms added to the Lagrangian in the Stückelberg language are higher order in total number of derivatives. It remains an open question if this feature does not lead to any pathologies. The problems with the break-down of black hole thermodynamics would arise in this approach as well: the field  $\chi$  generically has different propagation velocity compared to the graviton (helicity-2 excitation), thus violating the Lorentz invariance at low energies. It is unclear at present if and how these problems can be resolved.

Finally, we mention another interesting outcome of our study. We observed that the equation of motion for the Stückelberg field in Hořava gravity has a peculiar structure. This equation is explicitly higher order in covariant derivatives. Consequently in a general coordinate frame it is higher order in time. Generically, this would imply the presence of additional ghost modes. However, this reasoning is incorrect in our case: there is a unique preferred frame where the number of time derivatives in the equation is drastically reduced. Solving the Cauchy problem in this frame requires less initial data than in a general frame. The difference can be traced to the intrinsically non-relativistic nature of the theory which implies that the two frames are physically inequivalent. Technically this manifests itself in the fact that the boundary conditions at spatial infinity are not equivalent in the two frames. We pointed out that the reduction of degrees of freedom in a preferred Cauchy slicing is generic for a wide class of higher order covariant derivative operators. This opens up the possibility to construct a new class of consistent higher derivative theories with equations of motion based on these operators. We leave the investigation of these issues for the future.

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## A WKB expansion for the extra mode

In this appendix we construct the solution of eq. (5.2) using the WKB method. The application of this method is legitimate in the case when the frequency  $\omega$  and momentum

$p$  of the field  $\chi$  are much larger than  $L^{-1}$ , where  $L$  is the characteristic space-time scale of the background. The leading-order approximation to the solution is obtained in the main text and is given by the plain wave (5.3) with the frequency related to the momentum by the dispersion relation (5.4). Our aim here is to obtain the order  $1/(pL)$  corrections to this solution.

We work locally in the vicinity of the point  $x_o$ , which without loss of generality we assume to coincide with the origin of the coordinate frame. By the 3-dimensional diffeomorphism we can make the spatial metric at this point flat, and its first derivatives vanish. This implies that the spatial covariant derivatives in eq. (5.2) can be replaced by ordinary ones in our approximation. We now use the following ansatz for the field  $\chi$ :

$$\chi \propto \exp \left[ -i\omega^{(0)}t + ip_ix^i - i\delta\omega t + i\frac{a}{2}t^2 + ib_itx^i + \frac{i}{2}c_{ij}x^ix^j \right]. \quad (\text{A.1})$$

Here  $\omega^{(0)}$  is the leading-order frequency (5.4) evaluated at  $x_o$ ;  $\delta\omega$  is the frequency shift; and  $a, b_i, c_{ij}$  are the coefficients in the Taylor expansion of the WKB phase at  $x_o$ . Note that if we want the ansatz (A.1) to represent a small correction to the plain wave solution (5.3) in the region  $t, x^i \ll L$ , we must require

$$\delta\omega \ll \omega^{(0)}, \quad a \sim \omega^{(0)}L^{-1}, \quad b_i, c_{ij} \sim pL^{-1}. \quad (\text{A.2})$$

One proceeds by evaluating the derivatives of  $\chi$  appearing in eq. (5.2). Keeping the first subleading corrections one obtains,

$$\partial_i\dot{\chi} = [p_i\omega^{(0)} + p_i\delta\omega - p_iat - p_ib_jx^j + \omega^{(0)}b_it + \omega^{(0)}c_{ij}x^j + ib_i]\chi, \quad (\text{A.3})$$

$$\Delta^2\chi = [p^4 + 4p^2p_ib_it + 4p^2p_ic_{ij}x^j - 4ip_ip_jc_{ij} - 2ip^2c_{ii}]\chi. \quad (\text{A.4})$$

The last term in eq. (5.2) is already subleading because of the additional derivative of the background. Thus, it is enough to evaluate the corresponding  $\chi$ -derivative to the leading order

$$\partial_i\Delta\chi = -ip_ip^2\chi. \quad (\text{A.5})$$

Finally, the inhomogeneity of the background is taken into account by expanding the coefficients in (5.2) in a Taylor series,

$$\partial_i\bar{K} = \partial_i\bar{K}_o + \partial_i\partial_j\bar{K}_ox^j + \partial_i\dot{\bar{K}}_ot, \quad (\text{A.6})$$

$$\bar{N} = \bar{N}_o + \partial_i\bar{N}_ox^i + \dot{\bar{N}}_ot, \quad (\text{A.7})$$

where the quantities with the subscript ‘‘o’’ are evaluated at the origin  $x_o$ . We will omit this index in what follows.

The next step is to substitute the expressions (A.3)–(A.7) into eq. (5.2) and require the vanishing of the Taylor coefficients up to linear order in coordinates. This yields the following system of equations,

$$-2p_i\partial_i\bar{K}a + (2\omega^{(0)}\partial_i\bar{K} + 4\bar{N}p^2p_i)b_i + 2\omega^{(0)}p_i\partial_i\dot{\bar{K}} + p^4\dot{\bar{N}} = 0, \quad (\text{A.8})$$

$$-2p_i\partial_i\bar{K}b_j + (2\omega^{(0)}\partial_i\bar{K} + 4\bar{N}p^2p_i)c_{ij} + 2\omega^{(0)}p_i\partial_i\partial_j\bar{K} + p^4\partial_j\bar{N} = 0, \quad (\text{A.9})$$

$$p_i\partial_i\bar{K}\delta\omega + i\partial_i\bar{K}b_i - i\bar{N}(2p_ip_jc_{ij} + p^2c_{ii}) - 3ip^2p_i\partial_i\bar{N} = 0. \quad (\text{A.10})$$

Note that within the assumptions (A.2) the first term in eq. (A.9) is much smaller than the rest and we can neglect it. From this equation, the solution for  $c_{ij}$  reads:

$$c_{ij} = -\frac{V_i V_j}{U_k V^k},$$

where

$$\begin{aligned} U_i &= 2\omega^{(0)}\partial_i\bar{K} + 4p^2 p_i\bar{N}, \\ V_j &= 2\omega^{(0)}p^i\partial_i\partial_j\bar{K} + p^4\partial_j\bar{N}. \end{aligned}$$

It is easy to check that  $c_{ij}$  satisfies the estimate (A.2). We do not need to solve the system (A.8)–(A.10) in the full generality: any special solution is sufficient for our purposes. Thus we set  $b_i = 0$  and find  $a$  from (A.8). Finally, from eq. (A.10) we obtain the frequency shift

$$\delta\omega = \frac{i}{p^i\partial_i\bar{K}} \left\{ -\frac{2\bar{N}(p_j V^j)^2}{U_k V^k} - \frac{\bar{N}p^2 V_j V^j}{U_k V^k} + 3p^2 p_j \partial_j \bar{N} \right\}.$$

This expression is purely imaginary. Analyzing the order of magnitude of the terms entering into it one obtains the estimate (5.5) used in the main text.

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